

FORMAL NEIGHBORHOODS IN ARC SPACES

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ABSTRACT. The theorem of Greenberg-Kazhdan-Drinfeld describes the formal neighborhood of a closed arc. After giving a complete proof with examples, two possible versions for the relative case of the theorem are discussed. Each one is shown to hold for some classes of morphisms, including smooth morphisms and closed embeddings in smooth variety.

1. INTRODUCTION

The arc spaces have been introduced by J. Nash ([Nas95]) in an attempt to study the singularities of an algebraic variety X by attaching a geometric space intrinsically to X . This is the space of arcs X_∞ , which reflects the geometry of the singular locus without need to take resolution of singularities. It was introduced also in an attempt to characterize the essential divisors over X , i.e. those divisors appearing birationally as irreducible components of the exceptional locus in any resolution of singularities of X . After showing that there is an injection between the components in X_∞ of the singular fiber, and the essential divisors over X , Nash proposed what is called now the Nash problem on arc spaces. The geometry of these spaces has been studied actively since then ([IK03], [Ish04], [ELM04], [Mus02], [Mus01], [DL99], [Reg95], [Reg06], [FdB12], [FdB12], [Pl  05], [LJRL99], [PPP06], [Ish09], [GP07], H.Mourtada - Jet schemes of rational double point surface singularities, Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., 2014, [EMY03], [Per07], [Spi90], [IW10], [Ish13], M. Leyton-Alvarez - Resolution de probleme des arcs de Nash pour une famille d'hypersurfaces quasi-rationelles, Anales de faculte des sciences de Toulouse, vol.XX, 3, 2011, [GSLJ96], [GSLJ97], [NS05], [Pet07], [dF13], [Voj07], [PS12], and others). An important result about the geometry of arc spaces is the theorem of Grinberg - Kazhdan - Drinfeld, describing the formal neighborhood of a closed arc in X_∞ .

In this article we are trying to review the known facts about, or related to, the formal neighborhoods in arc spaces. The second section represents the theorem of Grinberg - Kazhdan, originally proved

over \mathbb{C} ([GK00]). Drinfeld extended it over arbitrary field k , giving a purely algebraic proof in his paper (Drinfeld - On the Grinberg - Kazhdan formal arc theorem, ArXiv: math/0203263v1, see also C.Bruschek, H.Hauser - Arcs, cords and felts - six instances of the linearization principle, American Journal of Mathematics, 132, 2010). Our purpose is to give a complete proof with details and explanations, often missing in the original paper, with an example, generalizing slightly the one given by Drinfeld. In the third section we discuss two possible approaches to the relative case of the theorem.

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Notations: 1) X is a scheme of finite type over a field k , with singular locus X_{sing} ;
 2) X_∞ is the space of arcs of X , parameterizing all morphisms $\text{Spec}(K[[t]]) \rightarrow X$, K/k being an extension;
 3) FN_{γ_0} is the formal neighbourhood of an arc $\gamma_0 \in X_\infty$, i.e. $\text{Spf}(\widehat{\mathcal{O}_{X_\infty, \gamma_0}})$, the completion being taken with respect to the maximal ideal m_{γ_0} .
 4) As a topological space, $\text{Spec}(k[[t]]) = \{0, \eta\}$, with 0 the closed point, and η the generic one.
 5) D^∞ is the formal scheme defined by the ring $k[[x_i, i \in \mathbb{N}]]$ of all formal expressions $\sum_{i=0}^\infty f_i$, each f_i a homogeneous polynomial of degree i in the variables x_j . This ring is the completion of $k[x_i, i \in \mathbb{N}]$ by the $(x_i, i \in \mathbb{N})$ -adic topology, viewed as an object in the category of topological rings and continuous homomorphisms (from now on we write (x) for $(x_i, i \in \mathbb{N})$). So, it is complete local ring with maximal ideal m , with the topology of projective limit, and a base for the open neighborhoods of 0 given by the decreasing sequence of ideals I_n , consisting of all power series with $f_i = 0, i \leq n$. The defined linear topology is weaker than the m -adic one, and though the ring $k[[x_i, i \in \mathbb{N}]]$ is complete in it, it is not complete in the m -adic topology. Moreover, $(x)k[[x]] \subsetneq m$, due to the fact that $k[[x]]$ is a completion of a non Noetherian ring.

2. GRINBERG - KAZHDAN - DRINFELD THEOREM

In this section is given a complete proof of the main theorem over arbitrary field k . For it we need the following

Definition 2.1. *A test ring A is a local k -algebra with a unit (not necessary Noetherian), having residue field k , and nilpotent maximal ideal m (i.e. $m^n = 0$ for some $n \in \mathbb{N}$). Let Testrings_k be the category of test rings with local homomorphisms.*

A typical example is $k[t]/(t^n)$.

Theorem 2.2. *(Grinberg-Kazhdan-Drinfeld) If $\gamma_0 \in X_\infty \setminus (X_{\text{sing}})_\infty$ is a k -arc, there exists a scheme of finite type Y and a point $y \in Y(k)$ with formal neighbourhood $FN_y = \text{Spf } \widehat{\mathcal{O}}_{Y,y}$, so that the formal neighbourhood FN_{γ_0} is isomorphic to $\text{Spf } \widehat{\mathcal{O}}_{Y,y}[[z_i, i \in \mathbb{N}]]$, the completion of $\widehat{\mathcal{O}}_{Y,y}[z_i, i \in \mathbb{N}]$ with respect to the ideal $(z_i, i \in \mathbb{N})$. That is, $FN_{\gamma_0} \simeq FN_y \hat{\times} D^\infty$.*

We start with some remarks:

1) The ring $\widehat{\mathcal{O}}_{Y,y}[[z_i, i \in \mathbb{N}]]$ consists of all power series of the form $\sum g_n$, with g_n homogeneous polynomials in variables z_i of degree n with coefficients in $\widehat{\mathcal{O}}_{Y,y}$.

2) If $\gamma_0 : \text{Spec } k[[t]] \rightarrow X$, and X' is the closure of the irreducible component of X_{reg} containing $\gamma(\eta)$, then $X_{\infty, \gamma_0} = X'_{\infty, \gamma_0}$, so we can assume X to be reduced and irreducible. Also, because the claim is local we could take X to be affine as well. But the scheme Y does not need to be neither reduced nor irreducible in general (see Example 2.5).

3) If S is arbitrary (not necessarily Noetherian) scheme over k , and $p \in S(k)$, the formal neighbourhood FN_p could be defined by its “restricted” functor of points

$h_S : \text{Testrings}_k \rightarrow \text{Sets}$,

$A \mapsto FN_p(A)$, the set consisting of all morphisms $\text{Spec } A \rightarrow FN_p$.

The reason that the category Testrings_k is enough to define the functor of points is that the complete local ring $\widehat{\mathcal{O}}_{S,p}$ is a projective limit of test rings. That is, for any complete local ring R ,

$\text{Hom}(R, \widehat{\mathcal{O}}_{S,p}) = \lim_{\leftarrow n} \text{Hom}(R, \widehat{\mathcal{O}}_{S,p}/m_p^n)$, so one could apply Yoneda’s lemma in the category of complete separated local k -algebras with linear topology weaker than the m -adic topology, to the category of functors $\text{Testrings}_k \rightarrow \text{Sets}$. In particular, for a test ring (A, m_A) the

A -points of FN_{γ_0} correspond to the $A[[t]]$ -points on X , whose reduction modulo m_A is γ_0 , i.e. to the A -deformations of γ_0 .

Proof of Thm 2.2. Let X be reduced, irreducible of dimension n , embedded in \mathbf{A}^N .

Claim 1. When working with local properties of $\pi^{-1}(X_{sing})$, $\pi : X_\infty \rightarrow X$ being the canonical projection, we could take without loss of generality X to be a locally complete intersection, which may be reducible. Indeed, let $r = \text{codim } X$, and let the ideal of X be $I_X = \{f_1, \dots, f_s\}$. Define $F_i = \sum a_{ij} f_j$, $i = 1, \dots, s$, with a_{ij} being general coefficients in k , and let $M \subset \mathbf{A}^N$ be the ideal $I_M = (F_1, \dots, F_r)$ defined by the first r of them. Then the following hold:

- 1) any irreducible component of M has dimension n , so M is a complete intersection scheme;
- 2) $X \hookrightarrow M$ is a closed subscheme, and X and M coincide at the generic point of X , that is, on an open nonempty subset;
- 3) there is some r -minor of the Jacobian matrix of M not vanishing at η_X ;
- 4) $X_{sing} \subset M_{sing}$.

Clearly, the same properties will hold for any choice of r among the F_i 's.

Claim 2. There exists closed affine complete intersection scheme of finite type $X' \supset X$ of the same dimension such that $\text{Im}(\gamma_0)$ is not contained in $\overline{X' \setminus X}$.

Indeed, take L to be the index set of all the r -tuples (i_1, \dots, i_r) of distinct integers with $i_j \in \{1, \dots, s\}$, and let M_l , for $l \in L$, be the corresponding complete intersection scheme. If there is no such an X' as claimed, for all M_l we would have $\overline{\text{Im}(\gamma_0)} \subset \overline{M_l \setminus X}$, thus $\text{Im}(\gamma_0)$ is contained in their intersection. But $\overline{M_l \setminus X} \cap X \subset \text{Sing}(M_l)$, and $\bigcap_l \text{Sing}(M_l) = X_{sing}$, contradicting the choice of γ_0 .

For such an X' we have $FN_{\gamma_0}^X = FN_{\gamma_0}^{X'}$, because $\gamma_0(\eta) \in X_{reg}$, so without loss of generality we can replace X by X' .

Now X is supposed to be complete intersection affine variety, contained in $\text{Spec } k[x_1, \dots, x_n, y_1, \dots, y_r]$, and defined by equations

$p_i = 0, i = 1, \dots, r$. Also, $\gamma_0(t) = (x^o(t), y^o(t)) = (x_1^o(t), \dots, x_n^o(t), y_1^o(t), \dots, y_r^o(t))$ is not contained for any t in X_{sing} , defined by $\det(\partial p_i / \partial y_j) = 0$.

For A a test ring, let $\gamma = (x(t), y(t))$, with $x(t) \in A[[t]]^n$, $y(t) \in A[[t]]^r$, be an A -deformation of γ_0 , i.e. its reduction modulo $m \subset A$ is equal to γ_0 . Because $\text{Im}(\gamma_0) \not\subset Z(\det(\partial p_i / \partial y_j))$, not all

coefficients of the power series $\det(\partial p_i / \partial y_j)(\gamma_0)$ are 0. So we may apply:

Lemma 2.3. (*Weierstrass preparation theorem*) *Let (R, m) be complete local separated ring with respect to a linear topology, which is weaker than the m -adic topology, $f = \sum_i c_i t^i \in R[[t]]$, with not all $c_i \in m$. If d is the first index such that $c_i \notin m$, then we have unique representation $f = q \cdot u$, for some monic polynomial $q = t^d + \sum_{0 \leq l < d} a_l t^l \in R[t]$ of degree d , with $a_l \in m$ for all l , and $u \in R[[t]]$ invertible.*

Thus, $\det(\partial p_i / \partial y_j)(x(t), y(t)) = q(t)u(t)$, for some invertible $u(t) \in A[[t]]$, and $q(t) \in A[t]$ a monic polynomial of degree d whose reduction modulo m equals t^d . The degree d depends on γ_0 only, not on the choice of its deformation γ . We may assume $d \geq 1$ because, if $d = 0$ we can eliminate y , and the claim holds.

The idea of the proof is to consider q as a new unknown variable. Then all A -deformations of γ_0 are in one-to-one correspondence with the solutions of the following system of equations with unknowns $q \in A[t]$, $x \in A[[t]]^n$, $y \in A[[t]]^r$:

$$\begin{aligned} \text{(I)} \\ \det\left(\frac{\partial p_i}{\partial y_j}\right)(x, y) &= 0 \text{ mod } q, \\ p(x, y) &= (p_1, \dots, p_r) = 0. \end{aligned}$$

Here, if the first equation holds, $q^{-1} \det(\partial p_i / \partial y_j)$ is invertible because it is invertible modulo m and m is nilpotent.

Next, for any fixed $e \in \mathbf{N}$, consider the following system with unknowns $q \in A[t]$, $x \in A[[t]]^n$, $\bar{y} \in A[[t]]^r / (q^e)$, such that q is monic polynomial of degree d , $q = t^d \text{ mod } m$, $x = x^o \text{ mod } m$, $\bar{y} \text{ mod } m = y^o \text{ mod } t^e$:

$$\begin{aligned} \text{(II)} \\ \det\left(\frac{\partial p_i}{\partial y_j}\right)(x, \bar{y}) &= 0 \text{ mod } q, \\ p(x, \bar{y}) &\in \text{Im}\left(q^e \left(\frac{\partial p_i}{\partial y_j}\right)(x, \bar{y}) : \frac{A[[t]]^r}{qA[[t]]^r} \rightarrow \frac{q^e A[[t]]^r}{q^{e+1}A[[t]]^r}\right). \end{aligned}$$

The second condition makes sense, if one takes the Taylor expansion of $p(x, y)$ and noting that $p(x, \bar{y})$ is well defined modulo $\text{Im}(q^e (\frac{\partial p_i}{\partial y_j})(x, \bar{y}))$.

Moreover, it is equivalent to the equation $\hat{C}p(x, y) = 0 \text{ mod } q^{e+1}$, where \hat{C} is the adjoint matrix to $C = (\frac{\partial p_i}{\partial y_j}(x, y))$ (i.e. $C\hat{C} = \det(C).I_r$), with $y = y(t) \in A[[t]]^r$ in the pre-image of \bar{y} . Indeed, if $p(x, \bar{y}) = q^e C(x, \bar{y}).v$ for some $v \in A[[t]]^r / (q)$, then $\hat{C}p(x, \bar{y}) = q^{e+1}v$. Conversely, if $\hat{C}p(x, \bar{y}) = 0 \text{ mod } q^{e+1}$, there is some w such that $\hat{C}p(x, \bar{y}) = q^{e+1}w$, so $C\hat{C}p(x, \bar{y}) = Cq^{e+1}w$. Then for some invertible u , $p(x, \bar{y}) = Cq^e u w$,

that is $p(x, \bar{y}) \in \text{Im } q^e C$.

Furthermore, for any fixed $e \in \mathbf{N}$ the last condition in (II) is equivalent to the following equations:

$$\begin{aligned} p(x, \bar{y}) &= 0 \bmod q^e; \\ \hat{B}p(x, \bar{y}) &= 0 \bmod q^{e+1}, \text{ where } B = (\frac{\partial p_i}{\partial y_j}(x, \bar{y})). \end{aligned}$$

Both come from the second condition in (II), and the second equation makes sense once the first one holds. In this way the system (II) is equivalent to the following system which does not need any choice of $y \in A[[t]]^r$ such that $y \bmod q^e = \bar{y}$, and $x(t)$ is relevant up to $\bar{x} = x \bmod q^{e+1}$:

$$\begin{aligned} \text{(III)} \\ \det(\frac{\partial p_i}{\partial y_j})(x, \bar{y}) &= 0 \bmod q, \\ p(x, \bar{y}) &= 0 \bmod q^e; \\ \hat{B}p(x, \bar{y}) &= 0 \bmod q^{e+1}, \text{ where } B = (\frac{\partial p_i}{\partial y_j}(x, \bar{y})). \end{aligned}$$

Lemma 2.4. *For any e , the natural map from the set of solutions over A of system (I) to the set of solutions over A of system (II) is bijective.*

Proof. Let $c \in \mathbf{N}$ be the minimal number such that $m^c = 0$; we will prove the lemma by induction on c . If $c = 1$, i.e. $m = 0$, $A = k$, this holds because the both systems have one solution. Let $c \geq 2$, and suppose the claim holds for $c - 1$, that is, there exists $\tilde{y} \in A[[t]]^r$ with $\tilde{y} \bmod q^e = \bar{y}$ and $p(x(t), \tilde{y}(t)) \in m^{c-1}[[t]]^r$. This is the second equation of system I) taken over A/m^{c-1} . Such an \tilde{y} is unique modulo $q^e A[[t]]^r \cap m^{c-1}[[t]]^r$. To prove that the map between the set of solutions of the system (I) and the set of solutions of the system (II) is bijective we have to prove it has an inverse. That is, we have to find $z(t) \in q^e A[[t]]^r \cap m^{c-1}[[t]]^r$ s.t. $p(x, \tilde{y} - z) = 0$. As before, letting $C := (\frac{\partial p}{\partial y})(x(t), \tilde{y}(t))$, we would have $p(x, \tilde{y}) - C.z = 0$, because $m^{2c-2} = 0$ by the assumption $c \geq 2$. By the first equation in (II) and the argument following the system (I), $\det(C) = q.u$ for some invertible $u(t) \in A[[t]]$. This means that $z(t)$ will be unique if it exists, because $C.z = p(x, \tilde{y})$, and $\det(C) \neq 0$. By the second equation in (II), $p(x, \tilde{y}) \in q^e C.A[t]^r + q^{e+1}A[t]^r$. But $C.A[t]^r \supset C.\hat{C}.A[t]^r = \det(C)I_s.A[t]^r = q.u.A[t]^r$, so $q^{e+1}A[t]^r = q.q^e A[t]^r \subset q^e C.A[t]^r$. Thus $p(x, \tilde{y}) \in q^e C.A[t]^r$, that is, there is $z \in q^e A[t]^r$ such that $p(x, \tilde{y}) = C.z$. It remains to prove that $z \in m^{c-1}[t]^r$. By the induction we have $C.z = p(x, \tilde{y}) = 0 \bmod m^{c-1}$, and multiplying both sides by \hat{C} we have $\det(C)I_s.z = q.u.I_s.z = 0 \bmod m^{c-1}$. Thus $q.z = 0 \bmod m^{c-1}$,

and as q is monic, $z = 0 \pmod{m^{c-1}}$ as expected.

We are continuing the proof of Thm 2.2. From Lemma 2.3, the A -deformations of γ_0 are in one-to-one correspondence with the set of solutions of the system (II), and thus, with the set of solutions of the system (III) (for any fixed $e \in \mathbf{N}$). This latter set is, in fact defined by finite number of equations in finitely many variables, because $x(t)$ could be replaced with $\bar{x} = x(t) \pmod{q^{e+1}}$. Take $e = 1$, for example, so that $x(t) = q^2 \cdot \xi + \bar{x}$, where $\xi \in A[[t]]^n$, $\bar{x} \in A[t]^n$, $\deg(\bar{x}) < 2d$. We can consider \bar{x}, \bar{y}, ξ, q as new system of unknowns, replacing x, \bar{y}, q . Then (II) becomes a finite system of equations over k for \bar{x}, \bar{y}, q , with ξ not involved. By the remark about the restricted functor of points above, this proves that the formal neighbourhood of γ_0 is $FN_{\gamma_0} \simeq \text{Spf}(R[[z_i, i \in \mathbf{N}]])$, where R is a complete local Noetherian ring which defines the formal neighbourhood of a point on a scheme of finite type $y \in Y(k)$. FN_y is defined by equations including the variables \bar{x}, \bar{y}, q in terms of its functor of points. Thus for any k -algebra S , if $D := (\frac{\partial p}{\partial y})(\bar{x}, \bar{y})$, let

$$Y(S) := \{(q, \bar{x}, \bar{y}) : q \in S[t], \bar{x} \in S[t]^n/(q^2), \bar{y} \in S[t]^s/(q) :$$

$$\det(D) = 0 \pmod{q}, p(\bar{x}, \bar{y}) = 0 \pmod{q}, \hat{D} \cdot p(\bar{x}, \bar{y}) = 0 \pmod{q^2}\}.$$

Moreover, the point $y \in Y(k)$ will correspond to $(q = t^d, \bar{x} = x^0(t) \pmod{t^{2d}}, \bar{y} = y^0(t) \pmod{t^d})$.

The second factor is $D^\infty = \text{Spf } k[[z_i, i \in \mathbf{N}]]$, because $\xi \in A[[t]]^n$ has no restriction. This completes the proof.

Remark 2.5. 1) The number $d = \deg(q)$ in the proof of Thm. 2.2 is equal to $\text{ord}_{\gamma_0}(Jac_X)$, if the variables $y_j, j = 1, \dots, r$ are chosen in such a way among all variables, that the $r \times r$ minor defined by them in the Jacobian matrix of X has the minimal possible order among all such minors, which define locally the ideal of X_{sing} .

2) The scheme of finite type Y is not unique, because one could “enlarge” it replacing by $Y \hat{\times} D^n$ for any n .

Example 2.6. Let $\text{char } k = 0$ and $X : f(x_1, \dots, x_n) + x_{n+1}^s y = 0$ be a hypersurface in \mathbf{A}^{n+2} with coordinates (x_1, \dots, x_{n+1}, y) , for $f \neq 0$ a polynomial and $s \geq 1$ an integer. Suppose $\gamma^0 := (0, \dots, 0, t, 0) \in X_\infty$, viewed as an $(n+2)$ -tuple of formal power series, satisfying the equation of X . Any A -deformation γ of γ_0 is an $(n+2)$ -tuple of power series $(x_1(t), \dots, x_{n+1}(t), y(t))$ satisfying

$$(1) \ x_i(t) \in m[[t]], i = 1, \dots, n, \ y(t) \in m[[t]],$$

where $m \subset A$ is the maximal ideal of the test ring A . By Weierstrass division theorem, any A -deformation of $x_{n+1}^0(t) = t$ will be of the form $x_{n+1}(t) = (t - \alpha) \cdot u(t)$ for some $\alpha \in m$ and $u(t) \in A[[t]]$ invertible. Now,

given $\alpha, u(t), x_1(t), \dots, x_n(t)$, there will be at most one $y(t)$, satisfying (1), and it exists iff

$$(2) \begin{aligned} f(x_1(\alpha), \dots, x_n(\alpha)) &= f'(x_1(\alpha), \dots, x_n(\alpha)) = \dots \\ f^{(s-1)}(x_1(\alpha), \dots, x_n(\alpha)) &= 0 \end{aligned}$$

(the derivation is of power series with respect to t).

Indeed, if a solution $y(t)$ of $f(x_1(t), \dots, x_n(t)) + x_{n+1}^s(t)y(t) = 0$ exists then it is unique, and $(t - \alpha)^s$ divides $f(x_1(t), \dots, x_n(t))$. Conversely, if for an $\alpha \in m$, $f(\alpha) = f'(\alpha) = \dots = f^{(s-1)}(\alpha) = 0$, then $y(t)$ exists because $(t - \alpha)^s$ divides $f(x_1(t), \dots, x_n(t))$.

The system (2) defines a scheme of finite type Y with k -point $y = (0, \dots, 0)$.

3. THE RELATIVE CASE OF GRINBERG-KAZHDAN-DRINFELD THEOREM

Some questions about GKD theorem arise naturally. What is the geometric meaning of the scheme of finite type Y , appearing as first factor in it (M. Lejeune-Jalabert)? How would look like the relative case of the theorem (J. de Bobadilla)? In this section we will discuss two possible points of view, for morphisms with smooth domain, for smooth morphisms, and for closed embeddings in a smooth variety, among the others.

Remark 3.1. In the system (III) of the proof of Thm 2.2 we have coordinates \bar{x}, \bar{y}, q , and we take $e = 1$. Let X be embedded in \mathbb{A}^{n+r} , $n = \dim_P X$, $r = \text{codim}_P(X)$, and $d = \deg(q) = \text{ord}_{\gamma_0}(Jac_X)$. As the conditions of (III) give $d + rd + 2rd$ equations on the $2nd + rd + d$ coefficients of \bar{x}, \bar{y}, q , we have the following bounds for the dimension of Y :

$d(2n + r + 1) - d(1 + r + 2r) = d(2n - 2r) \leq \dim(Y) \leq d(2n + r + 1)$. We note that the bounds hold only for a scheme Y constructed in the proof of Thm.2.2. Would be interesting to understand the possible relation between any two schemes of finite type that could appear as the first factor in FN_{γ_0} .

Intuitively, the more singular is γ_0 , the bigger dimension has the scheme Y , chosen as the first factor in the formal neighborhood of γ_0 . If $d = \text{ord}_{\gamma_0}(Jac_X) = 0$, y could be eliminated, so $Y \simeq \text{Spf } k[[x_1, \dots, x_n]]$, and the first factor could be chosen to be $\text{Spec } k$. The following example shows that the opposite does not necessary hold.

Example 3.2. As a particular case of Ex.2.6, let $X : xy = 0 \subset \mathbb{A}^2$, and let $\gamma_0 = (t, 0)$. Because $\deg q = 1$, for the deformation γ we have

$x(t) = u(t)(t - \alpha)$, $y(t) = \beta$, with $u(t)$ invertible, and $xy = 0 \bmod q^2$. Thus $\beta = 0$, with no restriction for $\alpha \in m_A$, the maximal ideal of a test ring A , and for u . Using the argument in Ex.2.6, we conclude that in the formal neighborhood of FN_γ , the scheme of finite type Y could be chosen to be $\mathrm{Spf} k[[t]]$, or just $\mathrm{Spec} k$. That is, even when Y could be chosen smooth, it could happen $\gamma(0) \in X_{\mathrm{sing}}$.

Now suppose we have a morphism between algebraic varieties $f : X \rightarrow Y$, inducing morphism $f_\infty : X_\infty \rightarrow Y_\infty$ between the arc spaces. Given k -arcs $\gamma \in X_\infty \setminus (X_{\mathrm{sing}})_\infty$, $\delta \in Y_\infty \setminus (Y_{\mathrm{sing}})_\infty$ with $f_\infty(\gamma) = \delta$, there is induced morphism $\hat{f} : FN_\gamma \rightarrow FN_\delta$ between the formal neighbourhoods. By Thm 2.2, the first one is isomorphic to $\mathrm{Spf} \hat{\mathcal{O}}_{U,u}[[x_i, i \in \mathbf{N}]]$ for a scheme of finite type U and point $u \in U(k)$, and the second one is isomorphic to $\mathrm{Spf} \hat{\mathcal{O}}_{V,v}[[z_i, i \in \mathbf{N}]]$ for a scheme of finite type V and point $v \in V(k)$. Let $R_1 := \hat{\mathcal{O}}_{U,u}$, $R_2 := \hat{\mathcal{O}}_{V,v}$, and let $F : R_2[[z_i, i \in \mathbf{N}]] \rightarrow R_1[[x_i, i \in \mathbf{N}]]$ be the homomorphism, corresponding to $\hat{f} : FN_u \hat{\times} D^\infty \rightarrow FN_v \hat{\times} D^\infty$.

Proposition 3.3. *The morphism \hat{f} defines naturally a pair of morphisms, Φ between Noetherian formal schemes, and $\Psi : D^\infty \rightarrow D^\infty$ between the second factors in FN_γ, FN_δ .*

Proof. By restricting F we get homomorphism $\phi : R_2 \rightarrow F(R_2) \hat{\otimes}_k R_1$, which defines a morphism Φ between the corresponding formal schemes, which, composed with closed embedding, is a restriction of \hat{f} . The composition $\psi : k[[z_i, i \in \mathbf{N}]] \rightarrow R_2[[z_i, i \in \mathbf{N}]] \rightarrow R_1[[x_i, i \in \mathbf{N}]] \rightarrow k[[x_i, i \in \mathbf{N}]]$, where the first one is the natural inclusion, the second is F , and the last takes any power series modulo the maximal ideal $m_1 \subset R_1$, would define a morphism $\Psi : D^\infty \rightarrow D^\infty$.

In this way the pair of morphisms does not give much of geometric information, and in general it is not possible to obtain back \hat{f} from it. Would be preferable if \hat{f} could be uniquely determined by a pair of morphisms. The first one, Φ , is between FN_u , “extended” by the pre-image of FN_v , and FN_v . The second one, Ψ , is between the second factors D^∞ in the formal neighborhoods. More precisely, we would like that the corresponding homomorphism splits as $F = p \circ (\phi \hat{\times} \psi)$, where $\phi : R_2 \rightarrow F(R_2) \hat{\otimes}_k R_1$ is a homomorphism between complete local Noetherian k -algebras, and $\psi : k[[z]] \rightarrow k[[y]]$. Taking $p : F(R_2) \hat{\otimes}_k R_1[[y]] \rightarrow R_1[[x]]$ to be the natural multiplication will give

$\hat{f} = (\Phi \hat{\times} \Psi) \circ \tilde{p}$, where the natural closed embedding

$\tilde{p} : FN_\gamma \rightarrow \mathrm{Spf}(F(R_2) \widehat{\otimes}_k R_1) \hat{\times} D^\infty \simeq \mathrm{Spf} F(R_2) \widehat{\times}_k FN_\gamma$ is composed with $\Phi \hat{\times} \Psi : \mathrm{Spf} F(R_2) \widehat{\times}_k FN_\gamma \rightarrow FN_\delta$.

If this holds for any γ, δ as above, we could view it as the first version of Thm 2.2, which we call GKD theorem for morphisms.

For which classes of morphisms f could \hat{f} be represented in this way? We will prove it for morphisms with a smooth X , for étale and smooth morphisms, and for closed embeddings in smooth variety (all conditions satisfied locally in some neighborhoods of $P = \gamma(0)$ and of $Q = \delta(0)$).

We start with the case of smooth X .

Proposition 3.4. *For any $f : X \rightarrow Y$ with $P = \gamma(0) \in X_{\mathrm{reg}}$, the GKD theorem for morphisms holds for f . In particular, it holds for any resolution of singularities of Y .*

Proof. This time $F : R_2[[z_i, i \in \mathbb{N}]] \rightarrow k[[x_j, j \in \mathbb{N}]]$. We define $\phi : R_2 \rightarrow F(R_2)$, and we let $\psi = F \circ i$, where $i : k[[z_i, i \in \mathbb{N}]] \hookrightarrow R_2[[z_i, i \in \mathbb{N}]]$. Finally, the closed embedding is defined by the natural homomorphism $p : F(R_2) \widehat{\otimes}_k k[[x_j, j \in \mathbb{N}]] \rightarrow k[[x_i, i \in \mathbb{N}]]$. Obviously, F is fully determined by (ϕ, ψ, p) .

Another class of morphisms for which this first version of GKD theorem holds is that one of smooth morphisms. For the proof we need first the case of an étale morphism.

Proposition 3.5. *If f is an étale morphism the GKD for morphisms holds for f .*

Proof. If f is étale, the commutative diagram formed by $f : X \rightarrow Y$ and $f_\infty : X_\infty \rightarrow Y_\infty$ with the two canonical projections $\pi_X : X_\infty \rightarrow X, \pi_Y : Y_\infty \rightarrow Y$, is Cartesian. Thus $f_\infty : X_\infty \simeq \mathrm{Spec} \mathcal{O}_{X,P} \times Y_\infty \rightarrow Y_\infty$ is the projection, giving isomorphism $h : R_1[[x_i, i \in \mathbb{N}]] \rightarrow \hat{\mathcal{O}}_{X,P} \hat{\otimes} R_2[[z_i, i \in \mathbb{N}]]$. So the homomorphism F is the composition of h^{-1} with the natural inclusion $i : R_2[[z_i, i \in \mathbb{N}]] \rightarrow \hat{\mathcal{O}}_{X,P} \hat{\otimes} R_2[[z_i, i \in \mathbb{N}]]$. Define ϕ to be $R_2 \hookrightarrow \hat{\mathcal{O}}_{X,P} \hat{\otimes} R_2$, $\psi = id_{k[[z]]}$, and let $p = h^{-1}$. It is clear that they determine F completely.

Then, we get the case of smooth morphism.

Theorem 3.6. *The GKD for morphisms holds when f is smooth.*

Proof. In an open neighborhood $P \in U$, f could be represented as a composition of étale $g: U \rightarrow Y \times \mathbf{A}^n$, with the projection $pr_1: Y \times \mathbf{A}^n \rightarrow Y$. Now we need the following:

Lemma 3.7. *For any smooth V , and $pr_1: X \times V \rightarrow X$, if $\gamma \in (X \times V)_\infty \setminus (\text{Sing}(X \times V))_\infty$, $\delta \in X_\infty \setminus (X_{\text{sing}})_\infty$ are closed arcs such that $pr_{1,\infty}(\gamma) = \delta$, then GKD for pr_1 holds.*

Proof. We have $(X \times_k V)_\infty = X_\infty \times V_\infty$, so $\gamma = (\delta, \eta)$, and $FN_\gamma = FN_\delta \hat{\times} FN_\eta$. By Thm. 2.2, $FN_\delta = \text{Spf } R[[x_i, i \in \mathbf{N}]]$ for a complete local Noetherian ring R , thus $FN_\gamma = \text{Spf}(R[[x_i, i \in \mathbf{N}]] \hat{\otimes} k[[z_j, j \in \mathbf{N}]])$. Define $\phi: R \rightarrow R \hat{\otimes} R$, $\phi(q) = q \otimes 1$, take ψ to be the inclusion $k[[x_i, i \in \mathbf{N}]] \hookrightarrow k[[x_i, z_j, i, j \in \mathbf{N}]]$, and take $p: R \hat{\otimes} R \rightarrow R$ to be the natural homomorphism.

Continuing with the proof of Thm. 3.6., we apply Prop. 3.5 and Lem.3.7 to F . By taking $V = \mathbf{A}^n$, we get a composition of homomorphisms

$R_2[[u_i, i \in \mathbf{N}]] \rightarrow R_2[[u_i, v_j, i, j \in \mathbf{N}]] \rightarrow \mathcal{O}_{U,P} \hat{\otimes} R_2[[u_i, v_j, i, j \in \mathbf{N}]]$. Define $\phi: R_2 \rightarrow R_2 \hat{\otimes} \mathcal{O}_{U,P} \hat{\otimes} R_2$, $\phi(q) = q \otimes 1 \otimes 1$, and let $\psi: k[[u_i, i \in \mathbf{N}]] \rightarrow k[[u_i, v_j, i, j \in \mathbf{N}]]$ be the natural inclusion. Taking $p: R_2 \hat{\otimes} \mathcal{O}_{U,P} \hat{\otimes} R_2 \rightarrow \mathcal{O}_{U,P} \hat{\otimes} R_2$ to be the natural homomorphism will complete the proof.

Theorem 3.8. *For any closed embedding $f: X \hookrightarrow \mathbf{A}^n$ the GKD for morphisms holds.*

Proof. The induced morphism on arc spaces $f_\infty: X_\infty \hookrightarrow \mathbf{A}_\infty^n$ defines for an arc γ closed embedding $\hat{f}: FN_\gamma^X \hookrightarrow FN_\gamma^{\mathbf{A}^n}$. Following the proof of Thm 2.2 and taking $e = 1$, the system of equations (III) defines scheme of finite type (U, u) and the first factor $FN_u = \text{Spf } R$ of FN_γ^X . The corresponding to \hat{f} surjective homomorphism is $F: k[[z]] \rightarrow R[[x]]$. To define R one needs $N = d(2n+r+1)$ coordinates z_i in $FN_\gamma^{\mathbf{A}^n} \simeq \text{Spf } k[[z_1, \dots, z_N]][[Z]]$, where $Z = (Z_j, j \in \mathbf{N})$. So, there is natural surjective homomorphism between complete Noetherian local rings $k[[z_1, \dots, z_N]] \rightarrow R$, and we define $\phi: k[[z_1, \dots, z_N]] \rightarrow R \hat{\otimes} R$. The rest of the coordinates are the coefficients of power series, which are not under any restriction (see the end of the proof of Thm 2.2, p. 6). Let $\psi: k[[Z]] \rightarrow k[[x]]$ be the natural surjection, corresponding to the restriction of \hat{f} on D^∞ as second factor in FN_γ^X , and take

$p: R\hat{\otimes}R[[x]] \rightarrow R[[x]]$ to be the natural homomorphism.

Corolary 3.9. *The same holds for any smooth variety V , and $f: X \hookrightarrow V$ closed embedding.*

Proof. By the proof of Thm 2.2, in the system (I) the variables y_j now could be eliminated, so the variables $x_i, i = 1, \dots, n$ become free. Then the ring defining FN_γ^V is $k[[z]]$, and the surjective homomorphism $F: k[[z]] \rightarrow R[[x]]$ will define again a surjective $k[[z_1, \dots, z_N]] \rightarrow R$, where $n = \dim X$.

Proposition 3.10. *If $f_1: X \rightarrow Y, f_2: Y \rightarrow Z$ are morphisms, and $\gamma \in X_\infty \setminus (X_{\text{sing}})_\infty, \delta \in Y_\infty \setminus (Y_{\text{sing}})_\infty, \eta \in Z_\infty \setminus (Z_{\text{sing}})_\infty$ are closed arcs, such that $f_\infty(\gamma) = \delta, g_\infty(\delta) = \eta$. Let \hat{f}_1, \hat{f}_2 be the induced morphisms between their formal neighborhoods, with GKD theorem for morphisms holding for f_1 and γ, δ , and for f_2 and δ, η . Then it holds for $f_2 \circ f_1$ and γ, η .*

Proof. Suppose $F_1: R'[[y]] \rightarrow R[[x]]$ and $F_2: R''[[z]] \rightarrow R'[[y]]$ are the homomorphisms defined by \hat{f}_1, \hat{f}_2 , for complete local Noetherian rings R, R', R'' , and let (ϕ_i, ψ_i) define $F_i, i = 1, 2$, with appropriate surjections. Define ϕ to be the composition $R'' \rightarrow F_2(R'') \hat{\otimes} R' \rightarrow F_1(F_2(R'')) \hat{\otimes} F_1(R') \hat{\otimes} R$, which is a homomorphism between complete Noetherian rings, take $\psi = \psi_1 \circ \psi_2: k[[z]] \rightarrow k[[x]]$, and let p be the natural surjection as before. Then $F_1 \circ F_2 = p \circ (\phi \hat{\otimes} \psi)$.

Remark 3.11. 1) What happens if $\text{Im}(\gamma) \subset X_{\text{sing}}$? One could stratify X by $X_1: = X_{\text{reg}}$, then $X_2: = X_{\text{sing}} \setminus \text{Sing}(X_{\text{sing}})$, taking X_{sing} as a subscheme of X , and so on. This sequence will terminate after finite number of steps. Applying Thm 2.2 to each X_i in this sequence would represent the formal neighborhood of any closed arc in X_∞ , though taken as a point in a different subscheme of it.
2) An interesting question (J. de Bobadilla) would be if one could have a result similar to Thm 2.2 for (some class of) non-closed arcs on X_∞ as well.

Another possible approach to the relative case of GKD theorem goes as follows. This time the arcs considered are not points in the space of arcs X_∞ , but in the relative space of arcs $J_\infty(X/Y)$ ([Voj07]). First we remind some basic definitions and properties.

Let S be any ring, R, T any S algebras with $f : S \rightarrow R$ the natural homomorphism.

Definition 3.12. For $m \in \mathbb{N} \cup \{\infty\}$, a derivation of order m from R to T is a sequence (D_0, D_1, \dots, D_m) (respectively, (D_0, D_1, \dots)), where $D_0 : R \rightarrow T$ is an S -algebra homomorphism, and $D_i : R \rightarrow T$ is a homomorphism of abelian groups, such that for all $i > 0$

- 1) $D_i(f(a)) = 0$ for any $a \in S$;
- 2) $D_k(xy) = \sum_{i+j=k} D_i(x)D_j(y)$

The set of all derivations of order m is $\text{Der}_S^m(R, T)$.

Example 3.13. For an affine variety $X = \text{Spec } k[x]/(f)$, where $x = (x_1, \dots, x_n)$, $f = (f_1, \dots, f_s)$, the arc space is $X_\infty = \text{Spec } k[x^{(0)}, \dots, x^{(m)}, \dots]/(f, f', \dots, f^{(m)}, \dots)$, where $x^{(m)} = (x_1^{(m)}, \dots, x_n^{(m)})$. Define a derivation on the ring by $D(x_i^{(j)}) = x_i^{(j+1)}$, for any j , and any $i = 1 \dots n$, and let $D^{(p)} = D \circ \dots \circ D$, p times. Then $(D^{(0)} = \text{id}, D^{(1)}, \dots)$ is a derivation of order infinity of $k[x^{(0)}, \dots, x^{(m)}, \dots]/(f, f', \dots, f^{(m)}, \dots)$ into itself.

Definition 3.14. The R -algebra of Hasse - Schmidt derivations is $HS_{R/S}^m := R[x^{(i)}, x \in R, i = 0, \dots, m]/I$ (respectively, if $m = \infty$, $R[x^{(i)}, x \in R, i \in \mathbb{N}]/I$), where $x^{(i)}$ are variables, $x^{(0)} = x$ for any $x \in R$, and I is the ideal generated by the following relations: $(x + y)^{(i)} = x^{(i)} + y^{(i)}$, $f(a)^{(i)} = 0, i \geq 1$, $(xy)^{(k)} = \sum_{i+j=k} x^{(i)}y^{(j)}$, $x, y \in R, a \in S, i = 1, \dots, m$, for any $k \leq m$. The universal derivation $R \rightarrow HS_{R/S}^m$ is defined to be (d_0, \dots, d_m) , where $d_i(x) = x^{(i)} \mod I$.

Then it is easy to see that $HS_{R/S}^\infty = \lim_{\rightarrow m} HS_{R/S}^m$.

Proposition 3.15. i) $HS_{R/S}^m$ with the universal derivation represents the functor $\text{Der}_S^m(R, \cdot) : \text{Alg}_S \rightarrow \text{Sets}$.

ii) There is a natural bijection

$$\text{Der}_S^m(R, T) \rightarrow \text{Hom}_S(R, T[t]/(t^{m+1})), \text{ if } m < \infty, \text{ and}$$

$$\text{Der}_S^m(R, T) \rightarrow \text{Hom}_S(R, T[[t]]), \text{ for } m = \infty,$$

giving by i) natural bijection

$$\text{Hom}_S(HS_{R/S}^m, T) \rightarrow \text{Hom}_S(R, T[t]/(t^{m+1})), \text{ if } m < \infty, \text{ or}$$

$$\text{Der}_S^m(HS_{R/S}^m, T) \rightarrow \text{Hom}_S(R, T[[t]]), \text{ for } m = \infty.$$

This allows to formulate in case of schemes the following:

Theorem 3.16. There exists a sheaf $HS_{X/Y}^m$ of \mathcal{O}_X algebras, such that for any open sets $\text{Spec } A \subset Y$, and $\text{Spec } B \subset f^{-1}(\text{Spec } A)$, one has $\Gamma(\text{Spec } B, HS_{X/Y}^m) \simeq HS_{B/A}^m$, and these isomorphisms are compatible with localizations.

Definition 3.17. *The scheme of m jet differentials of X over Y is $J_m(X/Y) = \text{Spec } HS_{X/Y}^m$, and it represents the functor $\mathcal{S}ch_Y \rightarrow \mathcal{S}ets$, $Z \mapsto \text{Hom}_Y(Z \times \text{Spec } \mathbb{Z}[t]/(t^{m+1}), X)$ (respectively, $\text{Hom}_Y(Z \times \text{Spec } \mathbb{Z}[[t]], X)$).*

Suppose $f : X \rightarrow Y$ with f dominant, is a flat family. To avoid a component of the fiber of f to be singular on X , we assume $\dim X_{\text{sing}} < \dim X - \dim Y$ (in particular, $X_{\text{sing}} = \emptyset$ if $\dim X = \dim Y$). The relative arc space $J_\infty(X/Y)$ then could be viewed as the family of arc spaces on the fibers. We note that even having $P \in X_{\text{reg}}$, the fiber over Q could be singular at P .

For $\gamma \in J_\infty(X/Y)$ and δ closed arcs with $f_\infty(\gamma) = \delta$ as above, we have $\delta = \delta_Q$, the constant arc at $Q = f(P)$. The proof of Thm 2.2 for such γ remains valid, the coefficients being in $k(Q) \simeq k$.

Definition 3.18. *Let $FN_\gamma^{\text{rel}} \subset FN_\gamma$ be the formal neighborhood of the arc $\gamma \in J_\infty(X/Y)$.*

By Thm. 2.2 there are schemes of finite type, each with a k -point $(W, w), (U, u)$, such that the formal neighborhoods FN_w, FN_u are the first factors in $FN_\gamma^{\text{rel}}, FN_\gamma$ respectively. We note that (W, w) is a scheme of finite type over $\text{Spec } \mathcal{O}_{Y,Q}$. By Thm.2.1 in ([Voj07]), we get the sequence $J_\infty(X/Y) \rightarrow X_\infty \rightarrow Y_\infty \rightarrow Y$, in which the first morphism is closed embedding, the second morphism is f_∞ , and the last one is π_Y . The induced morphisms on formal neighborhoods, with the first one being again closed embedding, are

(1) $FN_\gamma^{\text{rel}} \rightarrow FN_\gamma \simeq FN_u \hat{\times} D^\infty \rightarrow FN_v \hat{\times} D^\infty \simeq FN_{\delta_Q}$. Let

$C_Q = \text{Spec}(\mathcal{O}_X \otimes_{\mathcal{O}_Y} k(Q))$ be the fiber over $Q = (c_1, \dots, c_l) \in Y$, so $\gamma \in (C_Q)_\infty$ (with some abuse of notation, because it is, in fact the image of γ in X_∞). From the proof of Thm. 2.2, the system of equations defining the first factor FN_w in FN_γ^{rel} includes all equations in system (I) defining FN_u , together with

$f_j(x_1(t), \dots, x_n(t)) = c_j, j = 1, \dots, r$, where $f = (f_1, \dots, f_r)$.

From L.2.3 there is a monic polynomial q' in this case, as we have monic polynomial q appearing in the system defining (U, u) . The Jacobian matrix for C_Q at P includes all the rows of the Jacobian matrix for X at P , with the same variables, and we conclude that $\deg q \leq \deg q'$.

For $\gamma \in J_\infty(X/Y)$, we have $\delta = \delta_Q$, and $\text{Im}(\delta) \not\subseteq Y_{\text{sing}}$, so $Q \in Y_{\text{reg}}$, giving from the proof of Thm 2.2, $FN_\delta = \text{Spf } k[[z_1, \dots, z_m]][[z]]$, where $m = \dim Y$. In this case the second group of the variables y_j could be eliminated (see p. 5), and the ring $k[[z_1, \dots, z_m]]$ defines both FN_v in (1), and FN_Q^Y . That is, for any closed arc $\gamma \in J_\infty(X/Y) \setminus J_\infty(X_{\text{sing}}/Y)$,

the first factor FN_w is a scheme over $FN_Q^Y \simeq \mathrm{Spf} k[[z_1, \dots, z_m]]$, which does not depend on Q .

Example 3.19. Take $X : x^2 - yz = 0$ as a flat family $X \rightarrow \mathbb{A}^1$, $(x, y, z) \mapsto x$, and let $\gamma = (0, t, 0)$. As we have seen in Ex. 2.6, the first factor in FN_γ could be taken as $\mathrm{Spec} k[t]/(t^2)$, and from Ex. 3.2, in FN_γ^{rel} the first factor could be taken as $\mathrm{Spec} k$.

Now we could formulate the second version of Thm. 2.2 for morphisms. Let $f : X \rightarrow X'$ be a morphism of Y -schemes, and let $f_\infty^{rel} : J_\infty(X/Y) \rightarrow J_\infty(X'/Y)$ be the corresponding morphism of relative arc spaces. Suppose $\gamma \in J_\infty(X/Y) \setminus J_\infty(X_{sing}/Y)$, $\delta \in J_\infty(X'/Y) \setminus J_\infty(X'_{sing}/Y)$ be closed arcs, with $f_\infty^{rel}(\gamma) = \delta$, and let $\hat{f}_{rel} : FN_\gamma^{rel} \rightarrow FN_\delta^{rel}$ be the induced morphism on relative formal neighborhoods, with the first factors defined by complete local Noetherian rings R_1, R_2 , respectively. Then for the corresponding homomorphism F_{rel} we are asking as before, if there is a couple of homomorphisms $\phi : R_2 \rightarrow F_{rel}(R_2) \hat{\otimes} R_1, \psi : k[[z]] \rightarrow k[[x]]$ so that with the natural surjection $p : F_{rel}(R_2) \hat{\otimes} R_1 \hat{\otimes} k[[x]] \rightarrow R_1[[x]]$ we would have $F_{rel} = p \circ (\phi \hat{\otimes} \psi)$.

When this holds for all γ, δ as above, we say that the relative version of GKD theorem holds for f . One could ask if the corresponding equivalents of Prop. 3.4, Thm. 3.6, and Thm. 3.8 are true in this version. As usual, all conditions are taken in affine open neighborhoods of P, Q , respectively.

Theorem 3.20. *Let $f : X \rightarrow X'$ be a morphism of Y -schemes, with $g : X \rightarrow Y, h : X' \rightarrow Y$, and $\dim(X_{sing}) < \dim X - \dim Y$, $\dim(X'_{sing}) < \dim X' - \dim Y$. Suppose $\gamma \in J_\infty(X/Y) \setminus J_\infty(X_{sing}/Y)$, $\delta \in J_\infty(X'/Y) \setminus J_\infty(X'_{sing}/Y)$ are k -arcs such that $f_\infty^{rel}(\gamma) = \delta$. The relative GKD holds for f if:*

- i) g is smooth morphism;
- ii) f is smooth morphism;
- iii) f is closed embedding, with h smooth morphism.

Proof. i) We want to show that there are homomorphisms (ϕ, ψ, p) as above, which will determine uniquely the homomorphism F_{rel} . If the first factor in FN_γ is $\mathrm{Spf} R$, and the first factor in FN_δ is $\mathrm{Spf} R'$, we have that $FN_\gamma^{rel} \hookrightarrow FN_\gamma, FN_\delta^{rel} \hookrightarrow FN_\delta$, taken with \hat{f}^{rel}, \hat{f} form a commutative diagram. The corresponding diagram of homomorphisms of rings contains then surjective homomorphism $H : R'[[z]] \rightarrow R_2[[v]]$ and $G : R[[x]] \rightarrow R_1[[u]]$. Here R, R_1 are the rings

defining the first factors in the formal and relative formal neighborhood of γ , and R', R_2 are the corresponding rings for δ . As g is smooth, its fibers are geometrically smooth, f is a morphism over Y , so we could apply Prop. 3.4.

ii) In this case, we have that f restricted to C_Q is smooth morphism, and \hat{f}_{rel} is defined by it. Then we obtain the claim by Thm.3.6.

iii) Now the restriction of f on the fiber C_Q is closed embedding in a smooth variety, namely, the fiber D_Q of h , and by Cor. 3.9 the claim holds.

Remark 3.21. It is easy to see that Prop. 3.10 also remains valid in this second version of GKD theorem.

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